RIGOROUS DETERMINATION OF THE PARAMETERS OF MICROSTRIP TRANSMISSION LINES*

Kenneth G. Black
General Electric Company
Schenectady, New York

Thomas J. Higgins
University of Wisconsin
Madison, Wisconsin

I Introduction

Statement of Problem

The purpose of this paper is to determine explicit expressions enabling calculation of the performance of the recently-introduced open-strip UHF transmission line, now rapidly coming into extensive use.

Importance of the Problem

A major problem stressed in the White Paper,[28] "The Fundamental Research Problems of Telecommunications" issued in 1948 by the Telecommunications Division of the British Post Office is that of obtaining wire transmission lines of better operating performance in those portions of the frequency spectrum devoted to UHF television, radio-relay, radar and like uses than was then available. The same need was, of course, recognized simultaneously in other countries as well. In consequence of the pressure of such need, intense activity in Great Britain, the United States,[1,2,3,4,8,11,26,27,29-35] Germany,[13] and other countries resulted in rapid development of several different types of wire-lines particularly useful in the UHF region.

Among those developed in this country the so-called strip transmission lines produced by the Federal Telecommunications Laboratories of the International Telephone and Telegraph Corporation have proved particularly useful for microwave work. Their low-loss characteristics, compactness of structure, ease of manufacture, and resulting reasonableness of cost render them particularly suited to low-cost mass-production techniques, especially for within-chassis microwave wiring where size reduction is important.

Strip-transmission lines comprise three major types: closed-strip, open-strip, and wire-above-ground plane. Of these, the open-strip line is the easiest to manufacture, its configuration being such that printed-circuit procedures or stripping of metal-plastic-metal laminations is easily applicable. In consequence, the open-strip line is much favored for low-cost general-purpose use where the somewhat larger radiation losses, than the closed-strip line, can be tolerated. A cross-section view of an open-strip is shown in Fig. 1-1.

Now by well-known theory,[6,10,13] determination of the electrical parameters prerequisite to calculation of the performance of a high-frequency communication line reduces — essentially — to determination of the associated capacitance, surface charge densities, and electric field distribution by use of basic electromagnetic theory. Now, although the dielectric slab separating the metal-strip conductors does not extend throughout all space, it is to be anticipated that, because of the large ratios of strip widths to strip spacing, essentially all of the surface charges on each of the two strips concentrates on the adjacent inner surfaces, whence in turn the electric field is essentially concentrated in the dielectric slab between the strips; and thus the capacitance, surface charge densities, and electric field distribution are essentially the same as though the whole of space is filled homogeneously with dielectric material. And in fact, such conjecture is confirmed by exhaustive experimental data stemming from an intensive study[1,2] of this probableness. On such basis, then, it ought to prove possible to effect determination of the mentioned desired electrical quantities enabling calculation of the operating performance of the open-strip line by well-known electromagnetic theory.

In principle such is possible. In actuality, explicit determination of the desired quantities for unequal-conductor-widths, open-strip lines can be expected to be extremely difficult to effect. In support of such remark we need only recall that explicit expression of these quantities for the geometrically-simpler, equal-conductor-widths, open strip-line, in form suited to engineering computation is available only through the chronologically successive efforts of such gifted workers in electrical theory as Maxwell,[14] Thomson,[24] Michell,[15] Bromwich,[5] Love,[12] and Moulton,[17] each of whom improved on and advanced the work of

* This paper is based upon a thesis supervised by Professor Higgins and submitted by Mr. Black to the faculty of the University of Wisconsin in August, 1953, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering.

his predecessors. The principal difficulty stems, of course, from the elliptic-function analysis required to obtain expression of the desired quantities. In consequence, it is not surprising that up to this time an explicit determination of the desired electrical quantities for the much more analytically-difficult problem of the unequal-conductor-widths open-strip line has not been effected.

Accordingly, it is for such reason that in their paper devoted to development of equations and charts enabling ready calculation of the operating performance of strip transmission lines Assadourian and Rimai[2] were, perforce obliged to use solutions for the two limiting cases of the open-strip line; namely, that where the two conductors are of equal widths and that where one conductor is of finite width (equal to that of the smaller conductor width of an actual open-strip line) and the other is of infinite width. Such calculation results, however, not in the desired actual determination of line performance, but of rather widely-separated upper and lower bounds to the performance. But more than this, Assadourian and Rimai, evidently lacking knowledge of the available rigorous solutions for these two cases used certain approximate equations which had been derived earlier by Maxwell and J.J. Thomson prior to the first formulation of the exact solution by Michell in 1894. These approximate equations yield values some 25 per cent greater than the correct values. Accordingly, they arrived at performance calculation not only between loose limits, but these limits are some 25 per cent greater than the correct values.

Obviously, therefore, what is most needed are explicit correct solutions enabling accurate calculations of the actual performance. The determination of precisely such expressions and illustration of their application through numerical example comprises the essential content of this paper.

Method and Scope of Solution

In Fig. 1-2 each of the two line segments is the cross-section of a conductor of the open-strip line, hence is the cross-section of a cylindrical equipotential surface. Accordingly, determination of the capacitance, surface charge densities and electric field distribution is a two-dimensional problem. Further, in view of the mirror symmetry of the cross-section in a plane perpendicular to and bisecting the conductors, we need only determine explicitly the field in one-half of the line, that in the other half being obtained by use of the mentioned symmetry. In view of this symmetry of field, and in that the lines CE and GA are flux lines, the strip-line can be considered as the conjunction of two strip-lines of "half-section" typified by the "polygon" ABCEFG. The conductors AC and EG of this half section are equipotential lines, the lines CE and GA are flux lines bounding the electric field.

Now by well-known theory, determination of the desired electrical quantities reduces to determination of the analytic function which maps the half-section of the strip-line on a rectangle in such fashion that equipotential lines AC and EG go into two opposite sides of the rectangle and the flux lines CE and GA go into the other two sides. This mapping can be done in two steps: mapping the half-section on an upper half-plane and subsequent mapping of the upper half-plane on the rectangle. Now the function mapping a rectangle on an upper half-plane is a well-known function. Accordingly, the mapping problem reduces to mapping of the half-section on the half-plane. In turn, however, the function mapping a rectilinear polygon such as the half-section on an upper half-plane is furnished by the corresponding Schwarz-Christoffel integral.[9] This integral is easily formulated and proves to be an elliptic integral of the third kind. All such integrals, except those of very simple integrands—not the case here—are notoriously difficult to integrate. However, after long and difficult study, integration was finally accomplished, yielding a general implicit expression of the desired complex potential function P in terms of Jacobian elliptic functions of the trigonometric and zeta types. This general expression contains three arbitrary constants, to be evaluated from known conditions stemming from the mapping. By successive, reiterated substitutions these constants are evaluated and hence explicit, implicit expression of the complex potential function is obtained.

The real and imaginary parts of P furnish the potential function Φ and flux function ψ characteristizing the electric field distribution. The negative gradient of the potential function evaluated at the surface of the conductors multiplied by a known constant, yields the surface charge densities. Accordingly, we have now to find only the expression for the capacitance. Effecting this in the usual fashion from the complex potential function results in an expression in terms of Jacobian elliptic functions of complex arguments. Unfortunately, no tables have been computed for direct evaluation of these functions; it is necessary to expand in terms of elliptic functions of real arguments, thus obtaining a very complicated expression for the capacitance. Moreover, as in the case in all solutions for the capacitance of transmission lines except those of very simple geometry wherefor the expression for the complex potential function can be reverted, the numerical values of capacitance for open-strip lines of specified dimensions must be calculated through use of a repeated cycle of computation. Since for the open-strip line three parameters occur—two conductor widths and the spacing—the calculation required for a specified line is both very long-drawn and, because of lack of suitable tables, numerically difficult to effect. However, the values of the capacitances of two strip-lines of specified dimensions have been calculated. Each of these values falls between the corresponding correct values for the above mentioned limiting cases of equal conductor widths and the widest conductor considered extended into an infinite plane, thus providing confirmation of the correctness of the general analysis and computation.

The mentioned computational labor suggests that an alternative, more facile means of calculating sufficient values of capacitance, enabling preparation of a set of universal curves yielding immediate determination of the capacitance, and thus —through multiplication by a known constant—the characteristic impedance of a strip-line of specified dimensions such as used in practice, is by use of the powerful method of subareas, which has earlier been successfully employed for ready numerical solution of other long unsolved difficult problems in transmission line calculation.[7,20,21,22] By means of this theory, and by the aid of digital computing equipment, some 18 values of capacitance were calculated and the desired set of universal curves plotted therewith.

Confirmative of the correctness of the subarea calculation in general, these curves are well distributed between the correct bounding curves for the previously mentioned two limiting cases of equal conductor widths and the widest conductor expanded into an infinite plane. Moreover, subarea calculation of the two strip-lines for which capacitances had earlier been calculated from the exact expression yields values in essential agreement with these exact values.

In conclusion, then, we note that the exact expressions for the field distribution and the capacitance and the set of universal curves obtained by subarea calculation both furnish solution of a difficult hitherto unsolved problem in electromagnetic theory and enable fairly rapid accurate calculation of the operating performance of the open-strip transmission line.

II Basic Theory

Inasmuch as the theory of calculation of the elctrical parameters of a high-frequency two-conductor line is well established in the literature (as in the book by Jordan,[10] and in, especially, the papers by Bucholz[6] and by Magnus and Oberhettinger[13]) it suffices in this paper only to outline the calculation in sufficient detail to make clear how the desired parameters of such a line can be computed.

Computation of Line Paramaters

Let the conductors be directed along the z-axis. Let $\Phi(x,y)$ be the associated electrostatic potential. Then $E_x = \partial\Phi/\partial x$, $E_y = -\partial\Phi/\partial y$, $H_y = (\varepsilon/\mu)^{1/2}\,\partial\Phi/\partial x$ and $H_x = -(\varepsilon/\mu)^{1/2}\,\partial\Phi/\partial y$. Accordingly, if n and t designate, respectively, the outward normal and counterclockwise tangent to a conductor surface, then $E_n = (\mu/\varepsilon)^{1/2} H_t$. In rationalized units the charge density σ at the surface of a conductor is $\sigma = \varepsilon E_n$. By Ampere's law $\oint H_t ds = J$, where the integral is taken around the perimeter of the conductor and J is the enclosed current. Let the current be assumed—as usual— to flow in a sheet of thickness $\delta = (2\rho/\mu\omega)^{1/2}$, and let j/δ designate the equivalent uniform longitudinal current density along the normal to the surface, such that $J = \oint j ds$. Then $\oint j ds = \oint H_t ds$, whence $j = H_t$. Also, $J = \oint H_t ds = \oint (\varepsilon/\mu)^{1/2} E_n ds = \oint (1/\mu\varepsilon)^{1/2}\,\varepsilon E_n ds = cQ$. Herein, $c = (1/\mu\varepsilon)^{1/2} \cong$ speed of light and Q is the charge on a unit length of the conductor.

Now the r-m-s power dissipation in a conductor in a unit length is

$$W = (1/2)\int (j/\delta)^2 \rho\, dv = (1/2)\oint (j/\delta)^2 \rho\delta\, ds$$
$$= (\rho/2\delta)\oint j^2 ds = (1/2)(\mu\omega\rho/2)^{1/2}\oint H_t^2 ds$$

The equivalent resistance R_c of the conductor, defined by $W = J^2 R_c/2$, is

$$\begin{aligned} R_c &= 2W/J^2 \\ &= J^{-2}(\mu\omega\rho/2)^{1/2}\oint H_t^2 ds \\ &= (cQ)^{-2}(\mu\omega\rho/2)^{1/2}\oint H_t^2 ds \\ &= (cQ)^{-2}(\mu\omega\rho/2)^{1/2}\oint (\sigma^2/\mu\varepsilon) ds \\ &= Q^{-2}(\mu\omega\rho/2)^{1/2}\oint \sigma^2 ds \end{aligned}$$

Obviously, the total resistance R of a unit length of the line is the sum of the individual conductor resistances.

The attenuation constant is defined by $\alpha = (R/2)(C/L)^{1/2}$. Since for high-frequency lines $LC = 1/c^2$, we have $\alpha = RCc/2$. Finally, the characteristic impedance $Z_o = (L/C)^{1/2}$ can be written as $Z_o = 1/cC$.

Accordingly, it follows from inspection of the expressions for R, α, and Z_o that if we have knowledge of the charge density σ and capacitance C, we can immediately calculate the desired values of R, α, and Z_o. Thus, the problem of calculation of the parameters of a broad-band high-frequency line reduces, essentially, to calculation of σ and C—and these are immediately determined, per the theory above, if the associated potential $\Phi(x,y)$ is known. We turn, then, to consideration of finding this quantity.

<u>Determination of Potential $\Phi(x,y)$</u>

Now because of the length and paralleiness of the conductors, the associated electric field is the same in each plane perpendicular to the axes of the conductors. Accordingly, the problem of determining $\Phi(x,y)$ is essentially a two-dimensional one. Recalling, as established in Section I, that the field distribution is to be found on the basis that the conductors are surrounded by a uniform nonpermeable medium of dielectric constant the same as that of the dielectric slab between the conductors, thus a medium of $\mu_r = 1$ and $\varepsilon_r = \varepsilon_d$, we have, from well-known theory, that determination of the potential function Φ is effected by finding that function which:

1. Satisfies Laplace's equation $\nabla^2\Phi = 0$ everywhere in the plane of the cross-section;
2. Is of specified constant values, say Φ_o and $-\Phi_o$, over the perimeters P_1 and P_2 of the cross-sections of the two conductors;
3. Vanishes at infinity except for an arbitrarily chosen constant, usually taken as zero.

The analytic difficulties attending the determination of Φ for a group of conductors of specified geometry are such that exact solution has been effected for relatively few conductor shapes and arrangements—even for the minimum of two conductors, the case of this paper. For in this last instance, determination of Φ depends, essentially, on finding the function which maps the area of the z-plane external to the conductor cross-section on the upper-half of a second plane, the t-plane, whence in turn this upper half-plane is to be mapped, through use of the well-known function t = sn w, on a rectangle in yet a third plane, the w-plane, in such fashion that the perimeters of the cross-section go into two opposite sides of the rectangle. But, as is well known, explicit determination of the first mapping function for a specified geometry is usually a difficult problem: in fact, the mapping functions are known for only a few of the many geometrical configurations of technical interest. We may note, however, that when the mapping function providing Φ cannot be found rigorously, it is yet possible to approximate Φ to any desired degree of accuracy, and thus solve the electrical problem accordingly, by theory pertinent to the powerful method of subareas.

Now although we shall have occasion to use subarea theory to effect a considerable number of numerical values, it is possible to obtain the mapping function which affords the rigorous solution of our problem; inasmuch as the cross-sections of the conductors are line-segments, the desired function mapping the area external to the conductors on an upper half-plane is afforded by the Schwarz-Christoffel transformation. Accordingly, we now set out the basis of this transformation, used in Section III, to obtain the mapping function underlying rigorous solution of the problem of this paper.

<u>The Schwarz-Christoffel Transformation</u>

The conformal transformation which maps the interior of any rectilinear polygon in the z-plane on the upper half of the t-plane, the perimeter of the polygon coinciding with the real axis of t, was advanced independently and more or less simultaneously by the German mathematician, H. A. Schwarz, and the Italian mathematician, E. B. Christoffel. Since the derivation is given at length in numerous sources[9] it suffices here to advance only the transformation proper: Thus,

$$z(t) = A\int^t (t-t_1)^{-\mu_1}(t-t_2)^{-\mu_2}\ldots(t-t_n)^{-\mu_n}\,dt+B \qquad (2\text{-}1)$$

Here $z = x + iy$ and t are respectively the variables of the z and t-planes. Assuming, first, that none of the images t_i of the vertices z_i of the polygon to lie at infinity in the t-plane, the $t_i (i = 1, \ldots, n)$ are positive or negative real numbers, such that in traversing the perimeter of the polygon in the positive direction (i.e., the interior of the polygon lying to the left), $\mu_i\pi$ is the angle turned through at the i^{th} vertex, counterclockwise turning being taken as positive. Finally, A and B are arbitrary constants, possibly complex, which determine respectively the orientation and location of the polygon in the z-plane relative to the x,y axes (Fig. 2-1).

Three of the vertices of the polygon can be assigned conveniently-chosen values of t_i on the real axis, provided only that these points t occur along the real axis in the same order as the corresponding vertices occur along the perimeter of the polygon. In a specific problem it is usually convenient to do so, thus fixing the values of A and B.

In that the mapping function z(t) is a function of its upper limit t, the transformation (2-1) can also be written in the form

$$dz/dt = A(t-t_1)^{-\mu_1}(t-t_2)^{-\mu_2}\ldots(t-t_n)^{-\mu_n} \qquad (2\text{-}2)$$

Now (2-2) can be expressed in the form

$$\frac{d}{dt}\left(\ln\frac{dz}{dt}\right) = \sum_{i=1}^{n} -\mu_i/(t-t_i) \qquad (2\text{-}3)$$

Accordingly, if one (or more) of the chosen t_i is a point at infinity, it follows (2-3) that the associated μ_i must equal zero; whence the corresponding factor (or factors) $(t-t_i)$ is to be omitted in (2-1).

III Transformations Between Planes

Relation Between Conductor Plane and Complex Potential Plane

To obtain the desired potential function $\Phi(x,y)$ we turn to explicit formulation of the complex potential function $P(z) = \Phi(x,y) + i\psi(s,y)$ which maps the upper half-section of the conductor-plane z of Fig. 3-1a on the rectangle in the P-plane of Fig. 3-1b. Now in Fig. 3-1a the line segment CE and the line GHIJA are flux lines, hence lines of constant flux function ψ; and the line segments ABC and EFG are equipotential lines, hence lines of constant potential function Φ. Let us, then, seek the function P(z) which carries ABC into a segment of the line $\Phi = \Phi_0$ and EFG into a segment of the line $\Phi = \Phi_0$ of the P-plane; and which carries COE into a segment of the line $\psi = 0$ and GHIJA into a segment of the line $\psi = \psi_0$ of the P-plane.

The values Φ_0 and ψ_0 are determined by the assigned potential difference between the conductors and the corresponding charges that appear on the conductors. Thus, let the potential difference between the conductors be V_0; and the charge per unit length on each conductor be Q_0; whence the number of flux lines per unit length originating on the half-conductor cross-section EFG and terminating on ABC is $4\pi Q_0/2 = 2\pi Q_0$. Then $\Phi_0 - (-\Phi_0) = 2\Phi_0 = V_0$ and $(\psi_0 - 0) = 2\pi O_0$; thus, $\Phi_0 = V_0/2$ and $\psi_0 = 2\pi Q_0$. With these relations established, we turn now to explicit formulation of the complex potential function P(z). This is effected in several steps.

Map of P-plane on w-plane

To obtain the desired mapping function, we first map the rectangle in the P-plane of Fig. 3-2a on the upper half of the t-plane of Fig. 3-2b. Accordingly, the mapping table, to employ this useful auxiliary tool introduced by Weber,[25] is as given in Table 3-1.

Introducing the values of Table 3-1 in (2-2) yields

$$dP/dt = A'(t+1/k)^{-\frac{1}{2}}(t+1)^{-\frac{1}{2}}(t-1)^{-\frac{1}{2}}(t-1/k)^{-\frac{1}{2}} \qquad (3\text{-}1)$$

Integration gives

$$P = A\int\frac{dt}{[(1-t^2)\ (1-k^2t^2)]^{\frac{1}{2}}} + B \qquad (3\text{-}2)$$

where A and B are constants. We now introduce the change of variable defined by

$$t = \operatorname{sn} w \qquad (3\text{-}3)$$

which maps the upper-half t-plane on the rectangle shown in the w-plane of Fig. 3-2c. From (3-3) we have

$$dt/dw = \operatorname{cn} w \operatorname{dn} w = [(1-t^2)(1-k^2t^2)]^{\frac{1}{2}} \qquad (3\text{-}4)$$

and thus

$$dw = dt/[(1-t^2)(1-k^2t^2)]^{\frac{1}{2}} \qquad (3\text{-}5)$$

Substituting accordingly in (3-2) gives

$$P = A\int dw + B \qquad (3\text{-}6)$$

and therefore

$$P = Aw + B \qquad (3\text{-}7)$$

To evaluate the arbitrary constants A and B we make use of the known values of corresponding points in the P-plane and the t-plane. From (3-3) we readily determine the values in the w-plane of Fig. 3-2c to be as in Table 3-2.

Substituting the values at points E

and C from Table 3-2 into (3-7) gives two equations, from which we easily find that

$$A = \Phi_0/K \text{ and } B = 0 \qquad (3\text{-}8)$$

Thus, from (3-7) we have

$$P = (\Phi_0/K)w \qquad (3\text{-}9)$$

<u>Map of z-plane on t-plane</u>

The next step is to map the z-plane on the t-plane. We effect this transformation in two steps, using an intermediate z'-plane as evidenced in Fig. 3-3. The values pertinent to the transformation are those of Table 3-3.

Introducing the values of Table 3-3 in (2-2) yields

$$dz/dz' = A'(z'-x_1')^{-\frac{1}{2}}(z'-C_1')^{1}(z'-x_2')^{-\frac{1}{2}} \times (z'-x_3')^{-\frac{1}{2}}(z'-C_2')^{1}(z'-x_4')^{-\frac{1}{2}} \qquad (3\text{-}10)$$

from which we obtain

$$\frac{dz}{dz'} = A'\frac{(z'-C_1')(z'-C_2')}{[(z'-x_1')(z'-x_2')(z'-x_3')(z'-x_4')]^{\frac{1}{2}}} \qquad (3\text{-}11)$$

This transformation maps the upper half of the z-plane of Fig. 3-3a on the upper half of the z'-plane of Fig. 3-3b, the half-conductors going into the two unequal segments of the real axis of the z'-plane. These two unequal segments of the z'-plane are mapped into two equal segments in the t-plane of Fig. 3-3c, by the well-known linear fractional transformation:

$$z' = \frac{z_0 t + \beta}{t + \theta} = z_0 + \frac{\alpha}{t + \theta} \qquad (3\text{-}12)$$

where z_0, α and θ are constants.

Now

$$\frac{dz}{dt} = \frac{dz}{dz'}\frac{dz'}{dt} \qquad (3\text{-}13)$$

and from (3-12) we have

$$dz'/dt = \alpha(t + \theta)^{-2} \qquad (3\text{-}14)$$

Substituting in (3-13) for (3-11) and (3-14) gives

$$\frac{dz}{dt} = A'\frac{(z_0+\frac{\alpha}{t+\theta}-C_1')(z_0+\frac{\alpha}{t+\theta}-C_2')}{(t+\theta)^2[(z_0+\frac{\alpha}{t+\theta}-x_1')(z_0+\frac{\alpha}{t+\theta}-x_2')]^{\frac{1}{2}}} \times \frac{1}{[(z_0+\frac{\alpha}{t+\theta}-x_3')(z_0+\frac{\alpha}{t+\theta}-x_4')]^{\frac{1}{2}}} \qquad (3\text{-}15)$$

and thus

$$\frac{dz}{dt} = A'\alpha\frac{[t+\theta+\frac{\alpha}{z_0-C_1'}][t+\theta+\frac{\alpha}{z_0-C_2'}]}{(t+\theta)^2[(t+\theta+\frac{\alpha}{z_0-x_1'})(t+\theta+\frac{\alpha}{z_0-x_2'})(t+\theta+\frac{\alpha}{z_0-x_3'})]^{\frac{1}{2}}} \times \frac{(z_0-C_1')(z_0-C_2')}{[(t+\theta+\frac{\alpha}{z_0-x_4'})(z_0-x_1')(z_0-x_2')(z_0-x_3')(z_0-x_4')]^{\frac{1}{2}}} \qquad (3\text{-}16)$$

By making substitutions,

$$C_1 + -\theta - \frac{\alpha}{z_0-C_1'}; \quad C_2 = -\theta - \frac{\alpha}{z_0-C_2'} \qquad (3\text{-}17)$$

$$\frac{\alpha}{z_0-x_1'} + \theta = -1/k; \quad \frac{\alpha}{z_0-x_4'} + \theta = 1/k \qquad (3\text{-}18)$$

and

$$\frac{\alpha}{z_0-x_2'} + \theta + -1; \quad \frac{\alpha}{z_0-x_4'} + \theta = 1 \qquad (3\text{-}19)$$

we obtain

$$\frac{dz}{dt} = \frac{A'\alpha(z_0-C_1')(z_0-C_2')k}{[(z_0-x_1')(z_0-x_2')(z_0-x_3')(z_0-x_4')]^{\frac{1}{2}}} \times \frac{(t-C_1)(t-C_2)}{(t+\theta)^2[(t+1)(t-1)(kt+1)(kt-1)]^{\frac{1}{2}}} \qquad (3\text{-}20)$$

Combining constants gives

$$\frac{dz}{dt} = A\frac{(t-C_1)(t-C_2)}{(t+\theta)^2[(1-t^2)(1-k^2t^2)]} \qquad (3\text{-}21)$$

<u>Map of z-plane on the P-plane</u>

If next we introduce in (3-21) the previously used transformation $t = \text{sn } w$ of (3-3) we can establish the desired function which maps the conductor z-plane on the complex potential P-plane. Now

$$\frac{dz}{dw} = \frac{dz}{dt}\frac{dt}{dw} \qquad (3\text{-}22)$$

Substituting in (3-22) from (3-4) and (3-21) and then replacing t by sn w from (3-3) gives

$$\frac{dz}{dw} = A\frac{(\text{sn } w-C_1)(\text{sn } w-C_2)}{(\text{sn } w+\theta)^2} \qquad (3\text{-}23)$$

Manipulation of the right-hand side, and use of the well-known identities

$$\text{sn}^2 w + \text{cn}^2 w = 1$$

and

$$\text{dn}^2 w + k^2\text{sn}^2 w = 1$$

yields

$$\frac{dz}{dw} = B\ dn^2 w + \frac{E'}{K'} - 1 \qquad (3\text{-}24)$$

$$+ \frac{(\theta + sn\ w)[-k^2 sn\ w\ cn^2 w - sn\ w\ dn^2 w] - cn^2 w\ dn^2 w}{(\theta + sn\ w)^2}$$

where B is an arbitrary constant. In the transition from (3-23) to (3-24) we have that

$$C_1 C_2 = \frac{E'\theta^2 - K'}{E' - k^2 K'\theta^2} \qquad (3\text{-}25)$$

and

$$C_1 + C_2 = \theta\ \frac{k^2 K' + K' - 2E'}{E' - k^2 K'\theta^2} \qquad (3\text{-}26)$$

After routine transformations of the Jacobian functions we obtain

$$\frac{dz}{dw} = B[dn^2 w + \frac{E'}{K'} - 1 + \frac{d}{dw}\frac{cn\ w\ dn\ w}{\theta + sn\ w}] \qquad (3\text{-}27)$$

Making use of the identity

$$EK' + E'K - KK' = \pi/2 \qquad (3\text{-}28)$$

in (3-27) gives

$$\frac{dz}{dw} = B[dn^2 w - \frac{E}{K} + \frac{\pi}{2KK'} + \frac{d}{dw}\frac{cn\ w\ dn\ w}{\theta + sn\ w}] \qquad (3\text{-}29)$$

Integrating (3-29) gives

$$z = B[\int^{w} zn^2 r\ dr - \frac{Ew}{K} + \frac{\pi w}{2KK'} + \frac{cn\ w\ dn\ w}{\theta + sn\ w}] + B_0 \qquad (3\text{-}30)$$

where B_0 is an arbitrary constant. Integrating in (3-30)

$$z = B[\ zn\ w + \frac{\pi w}{2KK'} + \frac{cn\ w\ dn\ w}{\theta + sn\ w}] + B_0 \qquad (3\text{-}31)$$

where zn(w) denotes Jacobi's zeta function.

To determine B and B_0 we make use of the known values of the points E and C in the z-plane and w-plane as given in Table 3-4, through coordination by the known values in the t-plane.

Substituting these values in (3-31) gives two equations, simultaneous solution of which yields

$$B_0 = 0; \qquad B = 2hK'/w \qquad (3\text{-}32)$$

Substituting (3-32) in (3-31) gives

$$z = \frac{2hK'}{\pi}[\ zn\ w + \frac{\pi w}{2KK'}\ \ \frac{cn\ w\ dn\ w}{\theta + sn\ w}] \qquad (3\text{-}33)$$

Finally, we note that substitution of

$$w = (K/\Phi_0)P \qquad (3\text{-}34)$$

from (3-9), into (3-33) would yield the desired expression for the complex potential function P. However, inasmuch as the resulting expression is implicit, rather than explicit, it is preferable to utilize (3-33) and (3-34) as parametric equations, in w, linking z and P.

IV Calculation of Capacitance

General Equation for the Capacitance

A general expression for the capacitance of the open-strip line is afforded by (3-9). Substituting the values for the points A and G of Table 3-2 into (3-7) gives

$$P = (2\pi Q_0/K')w \qquad (4\text{-}1)$$

Elimination of P and w between (3-9) and (4-1) gives

$$\Phi_0 = 2\pi Q_0 K/K' \qquad (4\text{-}2)$$

The capacitance per unit length of the lines is equal to the ratio of the potential difference between the conductors to the charge per unit length of the conductors, thus

$$C = V_0/Q_0 = 2\Phi_0/Q_0 \qquad (4\text{-}3)$$

Substituting (4-2) into (4-3) gives

$$C = K'/4\pi K \text{ per unit length of line} \qquad (4\text{-}4)$$

Specific Calculation of the Capacitance

In (4-4) the ratio K'/K for a specifically dimensioned line cannot be assigned from knowledge of the dimensions. Rather, we must proceed as follows. First we make use of the values at the points B and F shown in Figure 3-2c. These points have coordinates

$$w_B = -K + iv_2 \qquad w_F = K + iv_1 \qquad (4\text{-}5)$$

where v_1 and v_2 are yet to be determined. Substituting these values, together with the corresponding values for B and F in the z-plane as given in Fig. 3-1a in (3-33) we obtain

$$h + ib_1 = \frac{2hK'}{\pi}\left\{zn(K+iv_1) + \frac{\pi}{2KK'}(K+iv_1) + \frac{cn(K+iv_1)dn(K+iv_1)}{\theta + sn(K+iv_1)}\right\} \qquad (4\text{-}6)$$

and

$$-h+ib_2 = \frac{2hK'}{\pi}\left\{zn(-K+iv_2) + \frac{\pi}{2KK'}(-K+iv_2) + \frac{cn(-K+iv_2)dn(-K+iv_2)}{\theta + sn(-K+iv_2)}\right\} \qquad (4\text{-}7)$$

Equating the imaginary parts of each member of (4-6) and (4-7) gives

$$\frac{b_1}{h} = \frac{2K'}{i\pi}\left\{zn(K+iv_1) + \frac{i\pi v_1}{2KK'} + \frac{cn(K+iv_1)dn(K+iv_1)}{\theta + sn(K+iv_1)}\right\} \tag{4-8}$$

$$\frac{bb_2}{h} = \frac{2K'}{i\pi}\left\{zn(-K(iv) + \frac{i\pi v_2}{2KK'} + \frac{cn(-K+iv_2)dn(-K+iv_2)}{\theta + sn(-K+iv_2)}\right\} \tag{4-9}$$

We recall (3-25) and (3-26), rewritten here for convenience,

$$C_1 C_2 = \frac{E'\theta^2 - K'}{E' - k^2 K'\theta^2} \tag{4-10}$$

and

$$C_1 + C_2 = \theta\frac{k^2K' + K' - 2E'}{E' - k^2K'\theta^2} \tag{4-11}$$

Finally, from (3-3) and (4-5) we have

$$C_1 = sn(K + iv_1) \tag{4-12}$$

and

$$C_2 = sn(-K + iv_2) \tag{4-13}$$

In (4-8) through (4-13) we have six equations in six unknowns. Accordingly, solution of these six equations yields the value of the six unknowns C_1, C_2, v_1, v_2, θ and k. These are of such form, and are so interrelated, that we cannot solve them directly for k, which would then allow calculation of the corresponding value of $K'(k)/K(k)$ and thus in turn of C. Rather, we must proceed inversely. That is, we must assume a value of capacitance; then ascertain the geometry of the corresponding line; then through comparison of this geometry and that of the actual line try to determine how best to assume a new value of capacitance; and thus by a repeated cycle of such computation eventually come upon the capacitance of the specified line. A systematic schedule for such procedure of calculation is given in the following section.

Procedure for Calculation of Capacitance

1. Assume a value of capacitance
2. Calculate K/K' from (4-4)
3. Obtain k from tables 16, 18, 23
4. Assume a value of θ and by means of (4-10) and (4-11) determine C_1 and C_2
5. Determine v_1 and v_2 from (4-12) and (4-13) and tables 16, 23
6. Substitute values so obtained into (4-8) and (4-9) and solve for the ratios b_1/h and b_2/h
7. If the values so obtained in step 6 are not those of line whereof the capacitance is desired—and obviously this will usually be the case—a new value of θ must now be chosen and the procedure repeated from step 4 onwards.
8. If eventually it is found that no value of θ will yield the correct geometry, a new value of capacitance C must be chosen and steps 1 through 7 repeated.
9. By a repeated cycle of computation based on steps 1 to 8, the desired C can eventually be obtained.

Obviously, in this schedule of calculation, as in any schedule used in trial and error, judicious choice of assumed values will shorten the labor of calculation.

In performing the actual calculation, the work involved will be considerably greater than evidenced by mere recital of the steps of the procedure. Thus, in step 3, it is normally necessary to interpolate in available tables of K/K', the number of forward differences to be used depending upon the desired accuracy of the calculation. Again, step 5 cannot be carried out directly because there are no pertinent tables of the elliptic functions for complex arguments. Accordingly, it is necessary to assume values for v_1 and v_2 in (4-12) and (4-13), carry out the calculation for C_1 and C_2, and repeat until the desired values of v_1 and v_2 are obtained. In this step also, it is necessary to interpolate values in the used tables of real elliptic functions. Such interpolation is a laborious and time consuming process; thus, in the latter of the two calculations, carried out in detail as discussed below, it proved necessary to use up to and including fifth differences in Gauss' forward formula for interpolation.

Finally, a very considerable handicap in effecting the schedule of computation is the lack of tables of elliptic functions tabulated for values of argument between 89° and 90°, as required for computation in the range of most technical interest—namely, whereof the ratios of b_1/h and b_2/h are greater than 2. No doubt it is because of this lack of tables that Palmer,[19] who computed a curve for the capacitance of a strip-line with conductors of equal width, terminated his curve at a b/h ratio of 2.

Rigorous Calculation of the Capacitance of Two Specially-Dimensioned Lines

To check the accuracy of the analysis as a whole, to evidence that the scheme of calculation advanced can be effected, and to gain insight into the actual labor involved, capacitances of two specifically-dimensioned lines were computed. Assuming

C = 0.12733 statfarads per centimeter and proceeding as mentioned in steps 1 to 6 of the previous section, a line of ratios

$$b_1/h = 1.2633 \quad b_2/h = 0.2959 \quad (4\text{-}14)$$

was obtained.

To ascertain by what factor the experience gained in the first computation would enable shortening the time required, calculation was carried through for a second line. A value of C = 0.275314 statfarads/centimeter yielded ratios of

$$b_1/h = 2.79183 \quad b_2/h = 1.80635 \quad (4\text{-}15)$$

The calculations pertinent to this example are to be found in Appendix I.

A quasi-corroboration of the correctness of these values is afforded as follows. A line of ratios b_1/h and b_2/h will have a value intermediate between those lines of equal width of conductors whereof $b/h = b_1/h$ and a second like line of $b/h = b_2/h$. For a line with conductors of equal width and a ratio of $b/h = 1.8$, Palmer's[19] curve yields C = 0.24 statfarads/centimeter. By extrapolating Palmer's curve to $b/h = 2.8$, a capacitance of 0.32 statfarads/centimeter is obtained. Averaging these two values gives 0.28 statfarads/centimeter. This value is essentially equal to the value of C = 0.27531478 obtained by rigorous calculation. Accordingly, while such agreement cannot be taken as an absolute check on the correctness of the rigorous calculation, it does indicate that no gross errors occur in calculation of the two values found.

Practical Calculation of Capacitance

Inasmuch as it required a total of thirty hours to calculate the geometry of the second conductor corresponding to an assumed value of C, it would require a prohibitive amount of time to carry out the repeated cycles of such computation necessary to obtain the capacitance of a line of specific geometry—even after gain of considerable experience in such calculation. Obviously, then, what is most desired is a set of universal curves, say a family of curves of C as a function of b_1/h for various values of b_2/h.

Points for these curves could be found by initial assumption of C and calculation of corresponding values of b_1/h and b_2/h as is done in the example of the previous section. However, where a plenitude of skilled computing assistance is lacking, as is often the case, a more feasible approach to computation of the desired values of capacitance is by use of the method of subareas. Determination of the capacitance by this theory entails only simple algebraic manipulation, and the numerical computation can be effected in a routine manner on available IBM equipment. Accordingly, we turn to setting out the theory underlying the mode of computation.

V Method of Subareas

Fundamental Theory

The essential theory is to be epitomized as follows. Taking first the case of two cylindrical conductors, let each be considered as comprised, either exactly or approximately, of longitudinal substrips of area A_i, of number $(i = 1, \ldots, n)$ for the first conductor and $(i = n + 1, \ldots, n + n')$ for the second, which are:

(i) Of such small "width" w_i (distance measured tangentially along the perimeter of cross-section) by comparison with the total length of the corresponding perimeter that the charge density σ_i is essentially constant over each subarea A_i;

(ii) Of such shape that assumption of uniform charge density σ_i enables simple calculation of potential Φ_i produced by the uniformly distributed charge (per unit length) $q_i = \sigma_i w_i$ on A_i;

(iii) Of such dimensions and shape that if the subarea A_i were alone in space, the potential Φ_{ij} produced by A_i over that part of shape which is actually occupied by another subarea A_j is essentially constant, and similarly for the potential Φ_{ii} produced by A_i over itself.

Under these assumptions calculation proceeds as follows. By (i), Φ_{ij} produced over A_j by the charge q_i on A_i is proportional to q_i, whence $\Phi_{ij} = k_{ij}q_i$. Hence by (i) and (iii) the total potential over A_j is

$$\Phi_j = \sum_{i=1}^{n+n'} \Phi_{ij} = \sum_{i=1}^{n+n'} k_{ij}q_i,$$

a linear equation in the $n + n'$ unknowns $q_i(i = 1, \ldots, n + n')$. Proceeding thus to form the total potential over each subarea yields the set of $n + n'$ equations

$$\Phi_j = \sum_{i=1}^{n+n'} k_{ij}q_i$$

$(j = 1, \ldots, n + n')$. A well-known theorem in electrostatic theory states that the potential is constant over a charged conductor whereon the charge is in equilibrium. Imposing this condition over each each of the two sets of subareas comprising two conductor surfaces at potentials Φ_0 and Φ_δ yields

$$\Phi_o = \sum_{i=1}^{n+n'} k_{ij} q_i \; (j=1, \ldots, n)$$

and

$$\Phi_\delta = \sum_{i=1}^{n+n'} k_{ij} q_i \; (j=n+1, \ldots, n+n').$$

An additional equation stems from the fact that inasmuch as the total charges

$$Q_o = \sum_{i=1}^{n} q_i \text{ and } Q_\delta = \sum_{i=n+1}^{n'} q_i$$

on the two conductors are of equal magnitude but of opposite signs, the algebraic sum of the charges on the individual substrips is zero, thus

$$\sum_{i+1}^{n+n'} q_i = 0.$$

Solving the total set of (n+n'+1) equations for the q_i in terms of $(\Phi_o - \Phi_\delta)$ yields a set of values for the charge q_i over the subareas. In turn, these furnish an approximate value the charge Q through

$$Q_o = \sum_{i=1}^{n} q_i \quad \text{or} \quad Q_\delta = \sum_{i=n+1}^{n+n'} q_i.$$

Finally, the approximate value of the desired capacitance follows from $C=Q/(\Phi_o-\Phi_\delta)$, wherein the q_i, and hence Q, are expressed in terms of $(\Phi_o-\Phi_\delta)$ which cancels out on taking the ratio. Obviously, knowledge of charge density σ over the two conductors is furnished by the known values of the q_i.

Possible Simplifications

It commonly happens that the actual number of equations that must be solved is less than n+n'. Thus, if—as is usual—the configuration of conductor cross-sections is one such that one or more lines of symmetry exist with respect to which one-half of the configuration is the virtual image of the other half, the charge densities at two image points are on the same or different conductors. If the latter is the case, the line of symmetry is also an equipotential line (a cross-section of the corresponding equipotential plane), and can be assigned the reference value $\Phi = 0$; whence it then follows that the two conductors are at equal and opposite potentials, thus $\Phi_o = -\Phi_\delta$. Obviously, use of the charge relationship reduces the number of equations to be solved to less than the number of subareas (n+n'); namely, to the number N of unknown q_i. Again, use of the potential relation—when it exists—enables replacement of Φ_δ by $-\Phi_o$ and simplifies the solution of the equations somewhat by enabling direct solution of the q_i in terms of Φ_o rather than $(\Phi_o-\Phi_\delta)$, thus eliminating a certain amount of otherwise necessary algebraic manipulation.

Application

It is evident that the capacitance and charge distribution can be obtained to any desired degree of accuracy by taking subareas of sufficiently narrow width. Of course, the labor involved in manual solution of the set N linear equations increases rapidly with N. However, as evidenced by two examples given in a later chapter of this paper, surprisingly accurate values of capacitance and charge distribution can be obtained by use of small N, particularly if the two-conductor system possesses one or more geometric symmetries. Moreover, the ease with which a set of equations of high N can be solved by automatic calculating equipment (say by IBM punched-card machines or a Consolidated Engineering Corporation linear equation solver) and the general availability of such equipment and accompanying experienced aids afford a ready means of effecting actual numerical solution to a high degree of accuracy if such is required in a particular problem.

In conclusion, it is to be noted that although in this paper attention is confined to a two-conductor system, obvious extension and application of the basic theory enables determination of the capacitances and charge distribution associated with an arbitrary number of electrified parallel cylindrical conductors: thus, of a high-frequency transmission line whereof each of the two major conductors is comprised of several paralleled subconductors.

Derivation of the Basic Subarea Equations

for Uniformly-Charged Substrips

We advance here the basic equations for uniformly-charge substrips which we shall use in computation of the open-strip-line in Section 6.

With respect to Fig. 5-1, let q' be the uniformly-distributed charge per unit length of line, E_r the radial component of the electric field intensity at radial distance r from the line charge, and $\Phi(r)$ the corresponding potential. Then, in unrationalized c.g.s. units

$$-\partial\Phi/\partial r = E_r = 4\pi q'/2\pi r = 2q'/r \quad (5\text{-}1)$$

and

$$\Phi = -\int_\infty^r E_r d_r = 2 \ln \infty - 2 \ln r)q' \quad (5\text{-}2)$$

With respect to Fig. 5-2, let q be the uniformly distributed charge per unit of length of strip (whence $\sigma = q/d$ is the uniform charge density), E_r the radial component of electric field intensity at radial distance r from the center of the incremental strip of width dx (located distance x from the center of the strip) due to the charge σdx on the incremental strip, and Φ the corresponding potential. Then by (5-2)

$$d\Phi = (2 \ln \infty - 2 \ln r)dx \qquad (5\text{-}3)$$

and in that

$$r = (R^2 + x^2 - 2Rx \cos \theta)^{\frac{1}{2}}$$

we have from (5-3) that

$$\Phi = 2\int_{-0}^{d/2}(q/d)[2 \ln \infty - 2 \ln(R^2+x^2-2Rx \cos \theta)^{\frac{1}{2}}]dx \qquad (5\text{-}4)$$

The following three cases are of particular interest:

1. If R = 0, we have from (5-4)

$$\Phi = (2 + 2 \ln \infty - 2 \ln d + 2 \ln 2)q \qquad (5\text{-}5)$$

$$= (A + 3.386)q \qquad (5\text{-}6)$$

wherein $A = 2 \ln \infty - 2 \ln d$.

2. If $\theta = 0^0$, we have from (5-4)

$$\Phi = [2 + 2 \ln \infty + d_0^{-1}(R-d_0)\ln(R-d_0) - d_0^{-1}(R+d_0)\ln(R+d_0)]q \qquad (5\text{-}7)$$

wherein $d_0 = d/2$.

3. If $\theta = 90^0$, we have from (5-4)

$$\Phi = [2 + 2 \ln \infty - \ln(R^2+d_0^2) - 2(R/d_0)\tan^{-1}(d_0/R)]q \qquad (5\text{-}8)$$

As a useful approximation, we note that if $R \gg d$, then (5-7) and (5-8) reduce to

$$\Phi = 2(\ln \infty - \ln R)q \qquad (5\text{-}9)$$

Fig. 5-3 evidences two conductors under which the approximation of (5-9) is valid. In subarea analysis of the open-strip line, such conductor is divided into a number of narrow substrips, when although the total conductors are located close together, most computation is effected under the condition of Fig. 5-3b, and hence (5-9) can effectively be utilized in calculation, thus considerably simplifying the computation otherwise necessary if it were necessary to use (5-8) *in toto*.

VI Results of Method of Subareas

First Approximation

If each strip were considered as comprised of one subarea, as in Fig. 6-1, then by (5-5) and (5-9) the approximate potential at the center of the upper strip is

$$\Phi_2 = (2 + 2 \ln \infty - 2 \ln d_2 + 2 \ln 2) + 2(\ln \infty - \ln D)(-q) \qquad (6\text{-}1)$$

$$= (2 + 2 \ln 2 - 2 \ln d_2 + 2 \ln D)q$$

The potential at the center of the lower strip is

$$\Phi_1 = (2 + 2 \ln \infty - 2 \ln d_1 + 2 \ln 2)(-q) + 2(\ln \infty - \ln D)q \qquad (6\text{-}2)$$

$$= (-2 - 2 \ln 2 + 2 \ln d_1 - 2 \ln D)q$$

The first approximation of the capacitance of the strip line is

$$C = q/(\Phi_2 - \Phi_1) = 1/[6.772 + 2 \ln(D/d_1 d_2)] \text{ statfarads/centimeter} \qquad (6\text{-}3)$$

Second Approximation

Consider each strip as comprised of two equal subareas, as in Fig. 6-2. Because of symmetry, it is necessary to find the potential at the center of only one subarea of each strip. The approximate potential at the center of the left subarea of the upper strip is

$$\Phi_2 = [2 + 2 \ln \infty - 2 \ln(d_2/2) + 2 \ln 2]q + q[2 \ln \infty - 2 \ln(d_2/2)] - q[2 \ln \infty - 2 \ln r_1] - q[2 \ln \infty - 2 \ln r_2]$$

$$= q[2 - 4 \ln(d_2/2) + 2 \ln 2 + 2 \ln r_1 + 2 \ln r_2] \qquad (6\text{-}4)$$

The approximate potential at the center of the left subarea of the lower strip is

$$\Phi = [2 + 2 \ln \infty - 2 \ln(d_1/2) + 2 \ln 2](-q) - q[2 \ln \infty - 2 \ln(d_1/2)] + q[2 \ln \infty - 2 \ln r_1] + q[2 \ln \infty - 2 \ln r_3] \qquad (6\text{-}5)$$

Hence

$$\Phi_2 - \Phi_1 = q[4 - 4 \ln(d_2/2) + 4 \ln 2 + 4 \ln r_1 + 2 \ln r_2 + 2 \ln r_3 - 4 \ln(d_1/2)] \qquad (6\text{-}6)$$

Accordingly, the second approximation is

$$C = 2q/(\Phi_2 - \Phi_1) \qquad (6\text{-}7)$$

Fourth Approximation

The second approximation evidences all details of the manner in which the computation of the capacitance is effected. In actuality, in calculating C for plotting the universal curves, only fourth approximations (8 subareas per strip) were used. Feng[7] has shown that such subdivision yields quite accurate values of C. Fourth approximation yields a set of ten simultaneous equations, of which one stems from the potential difference between the conductors, another stems from the fact that the algebraic sum of the charges on the two conductors is zero, and the other eight stem from formulation of the potential equations of the subareas. Thus for a d_1/D ratio of 8 and a d_2/D ratio of 2 as evidenced in Fig. 6-3, the following set of equations results:

$$\Phi = \Phi_2 - \Phi_1$$

$$0 = q_1 + q_2 + q_3 + q_4 - q_5 - q_6 - q_7 - q_8$$

$$0 = -\Phi_2 + 5.0396q_1 + 1.9616q_2 + 0.9400q_3 + 0.5754q_4 + 5.0695q_5 + 3.8091q_6 + 2.2229q_7 + 1.1931q_8$$

$$0 = -\Phi_2 + 1.9616q_1 + 5.7125q_2 + 2.7726q_3 + 1.9616q_4 + 5.1176q_5 + 3.8839q_6 + 2.2761q_7 + 0.8333q_8$$

$$0 = -\Phi_2 + 0.9400q_1 + 2.7726q_2 + 6.7342q_3 + 4.1588q_4 + 5.1500q_5 + 3.9339q_6 + 2.3253q_7 + 5.840q_8$$

$$0 = -\Phi_2 + 0.5754q_1 + 1.9616q_2 + 4.1588q_3 + 8.9314q_4 + 5.1661q_5 + 3.9589q_6 + 2.3536q_7 + 0.4613q_8$$

$$0 = -\Phi_1 - 5.0685q_1 - 5.1176q_2 - 5.1500q_3 - 5.1661q_4 + 0.5056q_5 + 3.5836q_6 + 4.6050q_7 + 4.9698q_8$$

$$0 = -\Phi_1 - 3.8091q_1 - 3.8839q_2 - 3.9339q_3 - 3.9589q_4 + 3.5836q_5 - 0.1674q_6 + 2.7726q_7 + 3.5834q_8$$

$$0 = -\Phi_1 - 2.2229q_1 - 2.2761q_2 - 2.3253q_3 - 2.3536q_4 + 4.6050q_5 + 2.7726q_6 - 1.1890q_7 + 1.3862q_8$$

$$0 = -\Phi_1 - 1.1931q_1 - 0.8333q_2 - 0.5840q_3 - 0.4613q_4 + 4.9698q_5 + 3.5834q_6 + 1.3862q_7 - 3.3862q_8 \qquad (6\text{-}8)$$

Solution of the ten equations of (6-8) yields

$$q_1 = 0.0678\Phi;\ q_2 = 0.0355\Phi;\ q_3 = 0.0313\Phi;\ q_4 = 0.0297\Phi;\ q_5 = 0.0211\Phi;\ q_6 = 0.0211\Phi;\ q_7 = 0.0443\Phi;\ q_8 = 0.0777\Phi. \qquad (6\text{-}9)$$

Now

$$C = Q/\Phi = 2(q_1 + q_2 + q_3 + q_4)/(\Phi_2 - \Phi_1)$$

Substituting accordingly in (6-10) yields

$$C = 0.3286 \text{ statfarads/centimeter} \qquad (6\text{-}11)$$

In similar fashion, values of the capacitance were calculated for strip-lines over a range of geometry to be encountered in practice. These values are tabulated in Table 6-1.

Universal Curves

Fig. 6-4, wherein the curves are plotted from the values of Table 6-1, furnishes the desired set of universal curves. These afford by inspection the capacitance of any open-strip line to be encountered in practice, thus obviate the necessity of lengthy computation such as discussed in Section IV.

Corroboration of Rigorous Theory and Subarea Computation

In addition to the capacitances tabulated in Table 6-1, the capacitances of the two strip lines of the illustrative examples of Section IV were also obtained by subarea calculation. Table 6-2 comprises tabulation of corresponding values.

As evidenced, the values of capacitance obtained by subarea calculation are in close agreement with the values calculated from the exact equation

$$C = K'/4\pi K \qquad (6\text{-}12)$$

This agreement not only evidences the correctness of the results stemming from both modes of computation, but reaffirms the considerable value of subarea theory as affording a simple, straightforward means of calculating the otherwise analytically-difficult problem in potential theory.

Critical Comment on Equations Now Used for Design

We now turn our exact solutions to investigation of the accuracy of the work of Assadourian and Rimai,[2] who formulated the only equations hitherto available for the design of open-strip transmission lines. As stated in the introduction, these authors do not calculate actual values of capacitance or characteristic impedance for the open-strip line, but only upper and lower bounds. The upper bound occurs when $b_1 = \infty$; the lower bound when $b_1 = b_2$. Their values for the upper bound are calculated from approximate equations given by Maxwell and Thomson. They do not specifically state the equations used for calculation of the values for the lower bound.

To check the correctness of their lower bound, we can use the values obtained for the case of $b_1 = b_2$ as calculated in an earlier section, or we can obtain exact values from the papers by Palmer[19] and Magnus and Oberhettinger.[3] We choose the latter. We can obtain exact values for the upper bound $b_1 = \infty$ by use of the obvious fact that the value of capacitance for such is twice the capacitance of an open-strip line of equal strip widths, $b = b_2$ and ratio $b/h' = b_2/2h$, as indicated in Fig. 6-5. Exact values of the upper and lower bounds, so computed and tabulated in Table 6-3, furnish the curves of Fig. 6-6. Here, as defined by Assadourian and Rimai,

$$Z_0/Z_\delta = b/4\pi hC \qquad (6\text{-}13)$$

Fig. 6-6 indicates that, as stated in Section I, the curves of Assadourian and Rimai yield values some 25 per cent too high—which error stems from the fact that, in virtue of the way they are derived, Thomson's and Maxwell's equations, employed by them, yield poor approximations.

Finally, it is of interest to evidence that the subarea values tabulated in Table 6-1 are well-encompassed by the exact upper and lower bounds. Such is evident in Fig. 6-7, whereof we have plotted the characteristic impedance, Z_0, rather than C, thus simulatneously obtaining curves useful in open-strip line design. For air,

$$\mu = 4\pi \quad 10^{-7} \text{henry/meter} \qquad (6\text{-}14)$$

$$\varepsilon = (1/36\pi) \times 10^{-9} \text{ farad/meter} \qquad (6\text{-}15)$$

and thus

$$Z_0 = (L/C)^{\frac{1}{2}} = (\mu\varepsilon)^{\frac{1}{2}}/C = 30/C \text{ ohms} \qquad (6\text{-}16)$$

where the value of C is in statfarads/centimeter length of line. The several values which fall slightly below the lower bound are for the more disparate plate-widths, as is to be expected. To obtain the more accurate values bringing them above the lower bound would require the use use of at least an eighth approximation, easily obtained if the corresponding necessary card decks are available.

Finally, we note, as is to be expected, that those values of Z_0 calculated from the rigorous values of C obtained in the two illustrative examples of Section IV fall between the upper and lower bounds, as they should.

VII Summary

1. Exact expressions for the capacitance per unit length of line, C, and for the complex potential P(x,y) of the open-strip UHF transmission line have been derived. These enable calculation of the electrical parameters of the line, as detailed in Section II.

2. However, as evident from the procedure given in Section IV and from the example presented in the Appendix, exact calculation of the characteristic impedance of a specially dimensioned open-strip line (from $Z_0 = 30/C$) while possible, is extremely lengthy and numerically laborious to effect, since computation must proceed in an inverse, rather than direct, calculation.

3. Accordingly, a set of universal curves, plotted for values obtained by subarea calculation, have been effected. These yield the characteristic impedance of open-strip lines over a range of parameters likely to be encountered in practice.

4. Comparison of the exactly-determined upper and lower boundary curves show that the design equations now in use not only give values within certain broad limits but that both limits are about 25 per cent too high.

5. Comparison of numerical labor involved in effecting computation of C by the rigorous expression $C = K'/4\pi K$ and by subarea calculation evidences the considerable usefulness of the latter method as affording values by routine procedures using IBM equipment.

6. The general correctness of both the exact calculations and the subarea calculations are confirmed by the essential identity of the values obtained in Sections IV and VI by both methods for two specially dimensioned lines.

Appendix I

Calculation Typical of Determination of Specifically-Dimensioned Line

According to step 1 of Section IV, we assume a value of capacitance, say C = 0.275314 statfarads/centimeter. Then from (4-4) we have

$$C = 0.275314 = K'/4\pi K \qquad (A\text{-}1)$$

which yields

$$K/K' = 4\pi C = 3.45970 \qquad (A\text{-}2)$$

Obverse interpolation in a table of K/K' gives k as

$$k = 0.0174524$$

and (A-3)

$$k^2 = 0.000304586$$

Direct interpolation in tables of K and E yields

$$K = 1.57092$$
$$K' = 5.43491 \quad \text{(A-4)}$$
$$E' = 1.00075$$

Per procedure stated in step 4 of Section IV, we next assume a value for the parameter θ; in this case preliminary investigation indicates as a reasonable choice

$$\theta = 167.664$$
$$\theta^2 = 28111.37 \quad \text{(A-5)}$$

Next, solution of (3-25) and (3-26) for C_2 gives

$$C_2[\theta \frac{k^2K'+K'-2E'}{E' - k^2} - C_2] = \frac{\theta^2E' - K'}{k^2\theta^2K'-E'} \quad \text{(A-6)}$$

Substituting from (A-3), (A-4), and (A-5) in (A-6), we obtain

$$C_2^2 + 12.6483C_2 - 617.7046 = 0$$

Solution of this quadratic equation gives

$$C_2 = -31.9650, 19.3166 \quad \text{(A-7)}$$

Because of the nature of (3-25) and (3-26), (A-7) yields solutions valid for both C_1 and C_2. Let, then,

$$C_1 = -31.9650 \quad \text{and} \quad C_2 = 19.3166 \quad \text{(A-8)}$$

C_1 and v_1 are related by (4-12), repeated here for convenience,

$$C_1 = sn(K = iv_1) \quad \text{(A-9)}$$

Now

$$sn(K+iv_1) = \frac{dn(v_1,k')}{cn^2(v_1,k')+k^2sn^2(v_1,k')} \quad \text{(A-10)}$$

Accordingly, the value of v_1 must be obtained by trial and error process: assuming a value for v_1, solving for C_1, making a second choice of v_1, solving for C_1, and so on. This calculation for v_1 proves to be very lengthy, because normally the value of v_1 required to satisfy (A-9) will lie between given values in the separate tables of dn, cn and sn; whence the trial of v_1 results in necessity of interpolation in each of the three tables. However, after repeated cycles, the value of v_1 found to satisfy (A-9) was

$$v_1 = 4.24695 \quad \text{(A-11)}$$

This same lengthy process, repeated for v_2, yields value

$$v_2 = 3.68366 \quad \text{(A-12)}$$

Examination of (4-8) and (4-9) evidences that we must yet obtain the value of $zn(K+iv_1)$ and $zn(-K+iv_2)$. Now

$$zn(K+iv) = i\left\{-zn(v,k') - \frac{\pi v_1}{2KK'}\frac{dn(v_1,k')sn(v_1,k')}{cn(v_1,k')} - \frac{k^2sn(v_1,k')dn(v_1,k')}{cn(v_1,k')[cn^2(v_1,k')+k^2sn^2(v_1,k')]}\right\} \quad \text{(A-13)}$$

From (A-13) we note that the remaining value to be calculated is $zn(v_1,k')$. No published tables of this function are available in the range accruing to the present problem. $zn(v_1,k')$ can be, and was, found through use of the relationship

$$zn(v_1,k') = E(v_1,k') - v_1\frac{E'}{K'} \quad \text{(A-14)}$$

where $E(v_1,k')$ is a well-tabulated function.

Finally, substitution of values so obtained into (4-8) and (4-9) yield

$$b_1/h = 1.80635 \qquad b_2/h = 2.79183 \quad \text{(A-15)}$$

If, as would be the case in practice, we were seeking the capacitance of a specifically-dimensioned line, we would now compare these values with that of the line; commonly, they would not agree and we would start anew with a second value of θ or C, and so repeat this lengthy cycle of computation until agreement was attained.

References

1. Arditi, M., "Experimental determination of the properties of microstrip components," Convention Record of the 1953 IRE National Convention. Inst. Radio Eng., Part 10, pp. 27-37, 1953; Elec. Comm., Vol. 30, pp. 283-293, 1953.

2. Assadourian, F. and Rimai, E., "Simplified theory of microstrip transmission systems," Proc. IRE, Vol. 40, pp. 1652-1657; Elec. Comm., Vol. 30, pp. 36-45, 1953.

3. Barrett, R. M., "Etched sheets serve as microwave components," Electronics, vol. 25, pp. 114-118, June 1952.

4. Barrett, R. M. and Barnes, M. H., " "Microwave printed circuits," Radio and TV News, vol. 46, pp. 16, 31, September 1951.

5. Bromwich, T. J. I'a., "Note on a condenser problem," Messenger of Math., vol. 31, pp. 184-192, 1902.

6. Bucholz, H., "Berechnung von Wellenwiderstand und Dampfung von Feldbild des vollkommenen Leitens her," Arch. Elektrotech., vol. 39, pp. 79-100, 202-215, 1948.

7. Feng, T. N., "Determination by the Method of Subareas of the Parameters of Broad-Band Lines and Cables Comprised of Strip, Rectangular-Tubular or Circular-Tubular Conductors," Ph. D. Thesis, University of Wisconsin, 1951.

8. Grieg, D. D. and Engelmann, H. F., "Microstrip—A New Transmission Technique for the Kilomegacycle Range," Proc. IRE, vol. 40, pp. 1644-1650, 1952; Elect. Comm., vol. 30, pp. 26-35, 1953; Proc. IRE, Australia, Vol. 15, pp. 9-13, 1954.

9. Higgins, T. J., "An epitomization of the basic theory of the generalized Schwarz-Christoffel transformations as used in applied physics," Jour. Appl. Phys., Vol. 22, pp. 365-366, 1951.

10. Jordan, E. C., "Electromagnetic Waves and Radiating Systems," Prentice Hall, New York, 1950.

11. Kostriza, J. A., "Microstrip components," Proc. IRE, vol. 40, pp. 1658-1663, 1952; Elec. Comm., vol. 29, pp. 46-54, 1952.

12. Love, A. E. H., "Some electrostatic distributions in two dimensions," Proc. London Math. Soc., Vol. 22, series 2, pp. 335-369, 1922-23.

13. Magnus, W. and Oberhettinger, F., "Die Berechnung des Wellenwiderstandes einer Bandleitung mit kreisformigen bzw. rechteckigem Auszenlaiterquerschnitt," Arch. Elektrotech., vol. 37, pp. 380-390, 1943.

14. Maxwell, J. C., "A Treatise on Electricity and Magnetism," vol. 1, 3rd Ed., Oxford University Press, London, 1892.

15. Michell, J. H., "A Map of the complex Z-function: a condenser problem," Messenger of Math., vol. 23, pp. 72-78, 1894.

16. Milne-Thomson, L. M., "Jacobian Elliptic Function Tables," Dover Publications, Inc., New York, 1950.

17. Moulton, H. F., "Current flow in rectangular conductors," Proc. London Math. Soc., vol. 3, series 2, pp. 104-110, 1904-05.

18. Oberhettinger, F. and Magnus, W., "Anwendung der Elliptischen Functionen in Physik and Technik," Springer-Verlag, Berlin, 1949.

19. Palmer, H. B., "The Capacitance of a parallel-plate capacitor by the Schwarz-Christoffel transformation," Trans. AIEE, vol. 56, pp. 363-367, 1937.

20. Reitan, D. K. and Higgins, T. J., "Calculation of the capacitance of a circular annulus by the method of subareas," Trans. AIEE, vol. 70, pp. 926-931, 1951.

21. Reitan, D. K. and Higgins, T. J., "Calculation of the electrical capacitance of a cube," Jour. Appl. Physics, vol. 22, pp. 223-226, 1951.

22. Reitan, D. K., "Determination of the Capacitance of Capacitors of Rectangular and Annular Geometry by the Method of Subareas," Ph. D. Thesis, University of Wisconsin, 1952.

23. Spencely, G. W. and R. M., "Smithsonian Elliptic Functions Tables," Smithsonian Institution, Washington, D. C., 1947.

24. Thomson, J. J., "Recent Researches in Electricty and Magnetism," Oxford University Press, London, 1893.

25. Weber, E., "Electromagnetic Fields, Theory and Applications," vol. 1, John Wiley and Sons, Inc., New York, 1950.

26. "Manufacture of microstrip," Electrical Communication, vol. 29, pp. 251-259, 1952.

27. "New Trends in Transmission Lines," The Aerovox Research Worker, vol. 22; December 1952.

28. "The Fundamental Research Problems of Telecommunications," D.S.I.R., H. M. Stationery Office, 1948.

TABLE 3-1

Point	A	C	O	E	G	H,J
P-plane	$-\phi_o + i2\pi Q_o$	$-\phi_o + i0$	0	$\phi_o + i0$	$\phi + i2\pi Q_o$	$0 + i2\pi Q_o$
angle $\mu_i\pi$	$\pi/2$	$\pi/2$	0	$\pi/2$	$\pi/2$	0
μ_i	1/2	1/2	0	1/2	1/2	0
t-plane	-1/k	-1	0	1	1/k	$\pm\infty$

TABLE 3-2

Point	A	C	E	G
t-plane	-1/k	-1	1	1/k
w-plane	-K + iK'	-K	K	K + iK'
P-plane	$-\phi_o + i2\pi Q_o$	$-\phi_o$	ϕ_o	$\phi_o + i2\pi Q_o$

TABLE 3-4

Point	E	C
z-plane	h	-h
t-plane	1	-1
w-plane	K	-K

TABLE 3-3

Point	A	B	C	E	F	G	H,J
z-plane	-h + i0	$-h + ib_2$	-h + i0	h + i0	$h + ib_1$	h + i0	$\pm\infty$
$\mu_i\pi$	$\pi/2$	$-\pi$	$\pi/2$	$\pi/2$	$-\pi$	$\pi/2$	2π
μ_i	1/2	-1	1/2	1/2	-1	1/2	2
z'-plane	x_4'	C_2'	x_3'	x_2'	C_1'	x_1'	$\pm\infty$

TABLE 6-1

b_1/b_2	b_2/h = 1	2	4	8
1	0.1656	0.2557	0.4337	0.7791
2	0.2006	0.3067	0.4914	0.8794
4	0.2272	0.3286	0.5236	0.9482
8		0.3404	0.5325	

TABLE 6-2

b_1/h	b_2/h	Exact Cap.	Approx. Cap.	% Error
1.2633	0.2959	0.1273	0.1256	1.34
2.7918	1.8066	0.2753	0.2715	1.38

TABLE 6-3

b_2/h	Z_o/Z_o' $b_1 = b_2$	$b_1 = \infty$
0.15	0.171	---
0.3	0.266	0.171
0.5	0.338	---
0.6	---	0.266
1.0	---	0.338
1.1	0.488	---
2.0	0.610	---
2.2	---	0.488
4	---	0.610
5	0.742	---
10	0.795	0.742

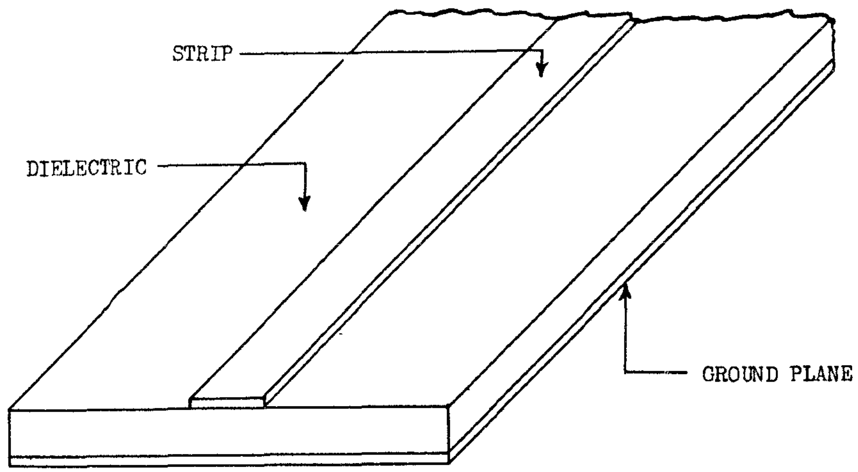

Fig. 1-1 - Typical open-strip transmission line.

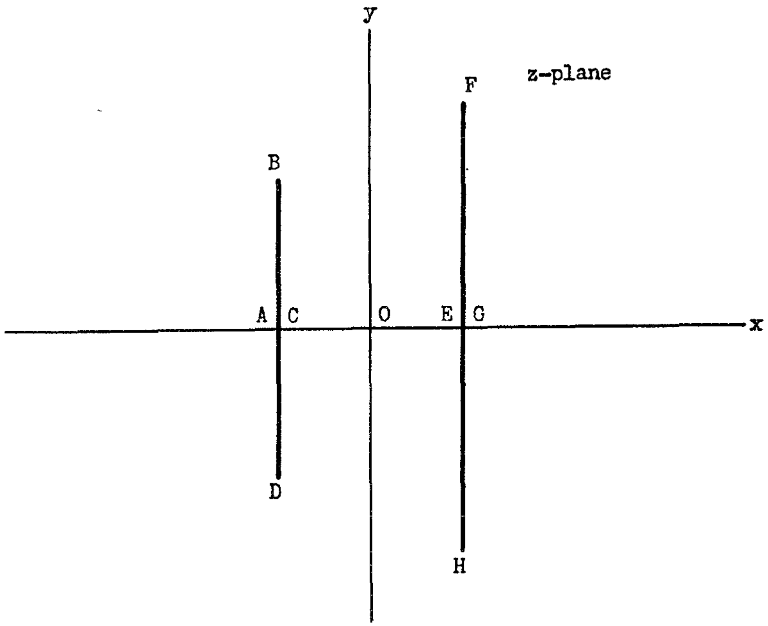

Fig. 1-2 - Cross-section of open-strip line.

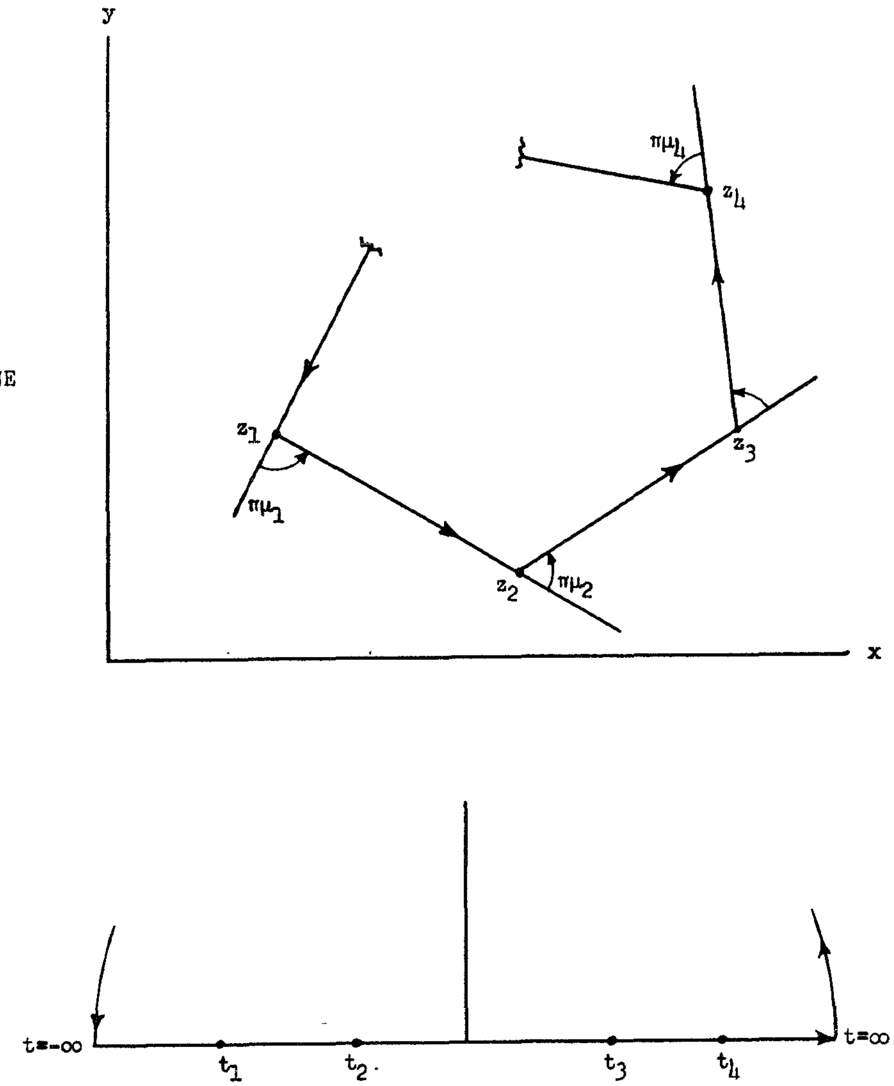

Fig. 2-1 - Correlation of z-plane and t-plane under Schwarz-Christoffel transformation.

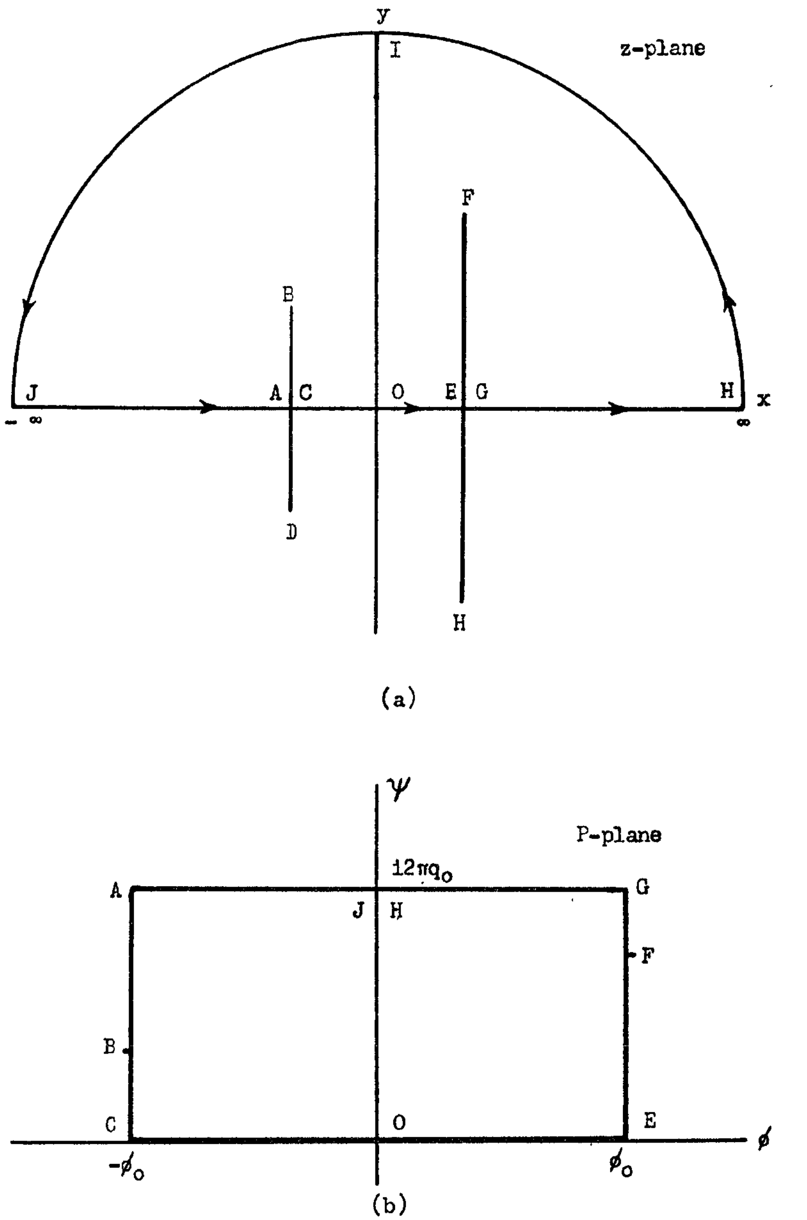

Fig. 3-1 - Correlation of z and P-planes under Schwarz-Christoffel transformation.

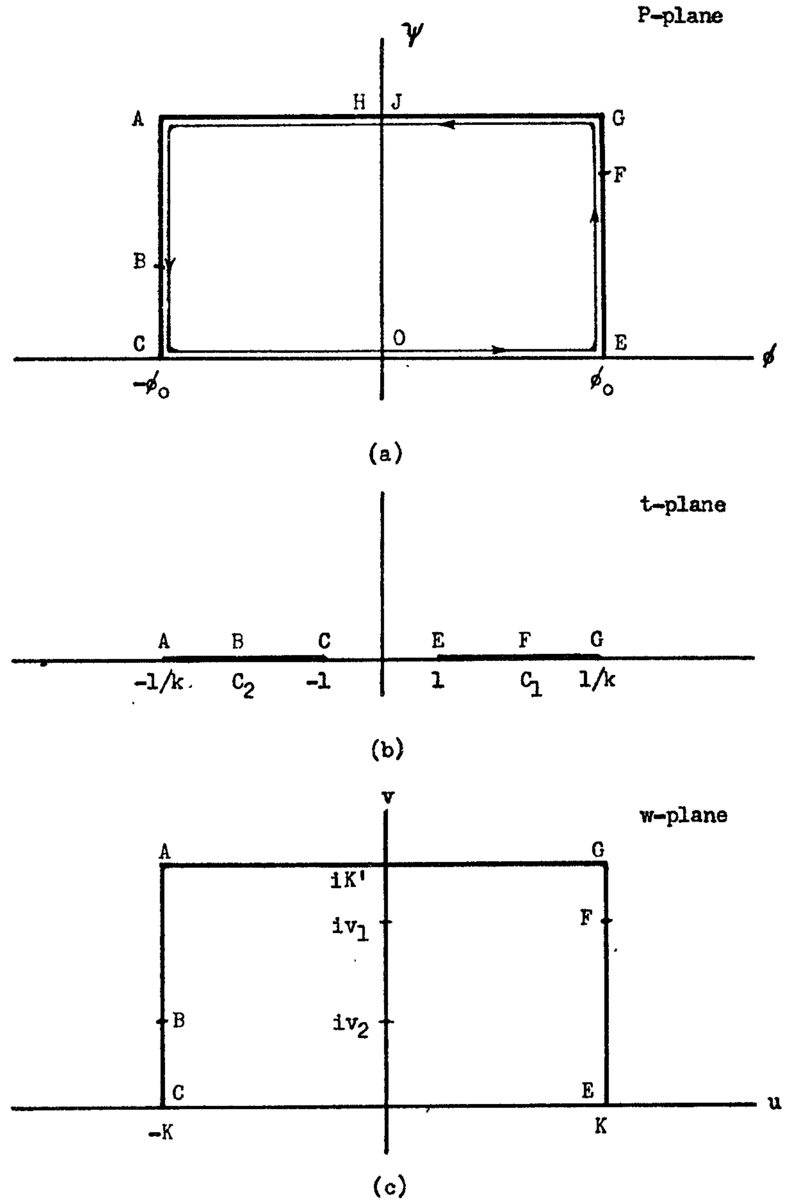

Fig. 3-2 - Correlation of P, t and w-plane under Schwarz-Christoffel transformation.

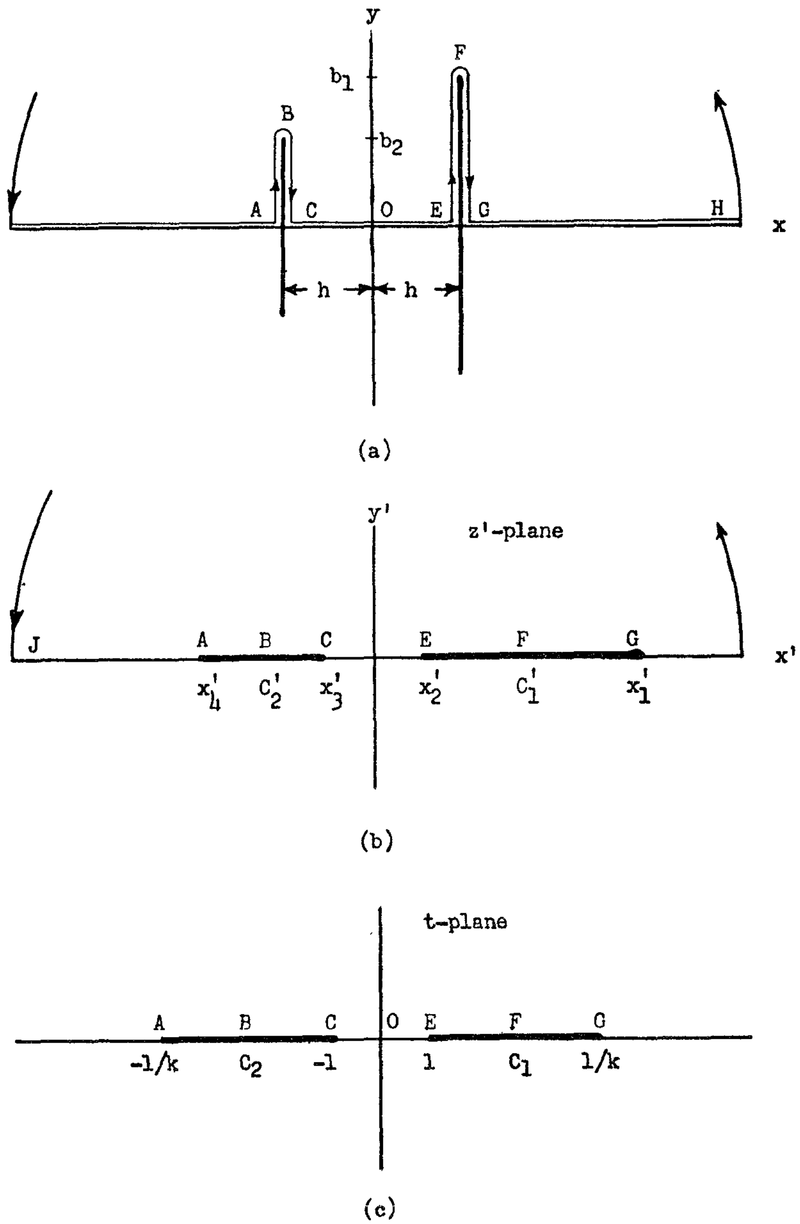

Fig. 3-3 - Correlation of z, z' and t-planes under Schwarz-Christoffel transformations.

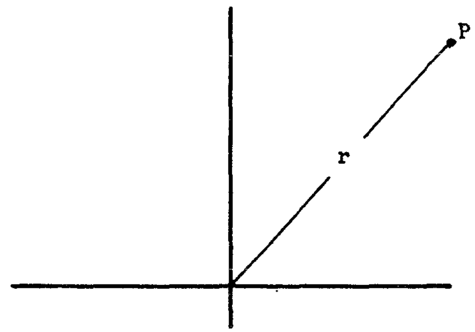

Fig. 5-1 - Potential due to uniformly charged strip.

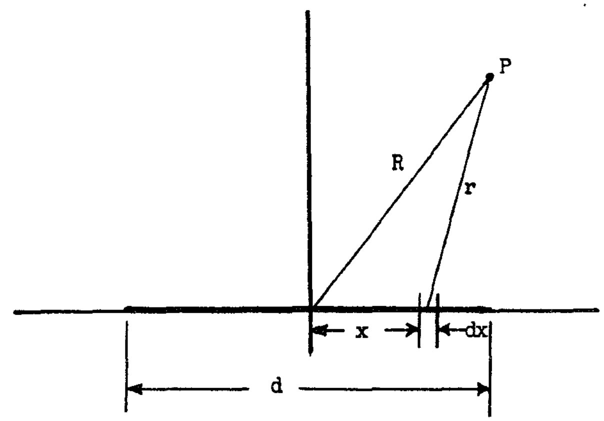

Fig. 5-2 - Potential due to uniformly charged incremental strip.

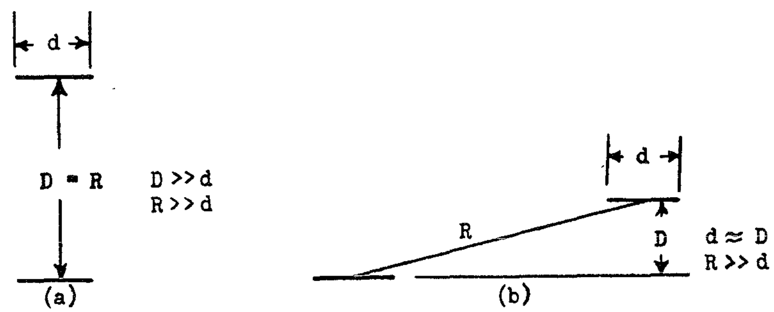

Fig. 5-3 - Geometries under which equation (5-9) is valid.

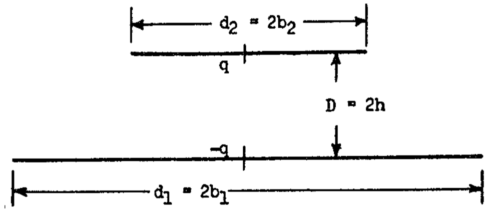

Fig. 6-1 - Subarea division for first approximation.

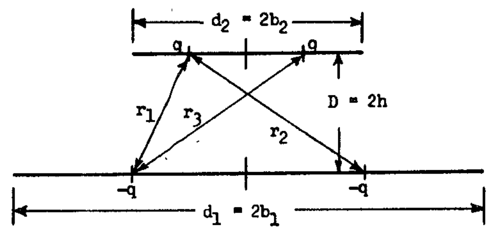

Fig. 6-2 - Subarea division for second approximation.

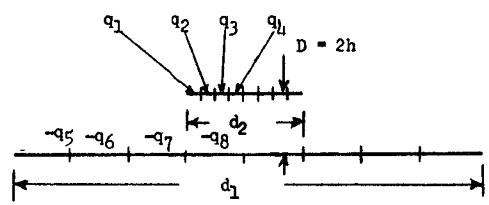

Fig. 6-3 - Subarea division for fourth approximation.

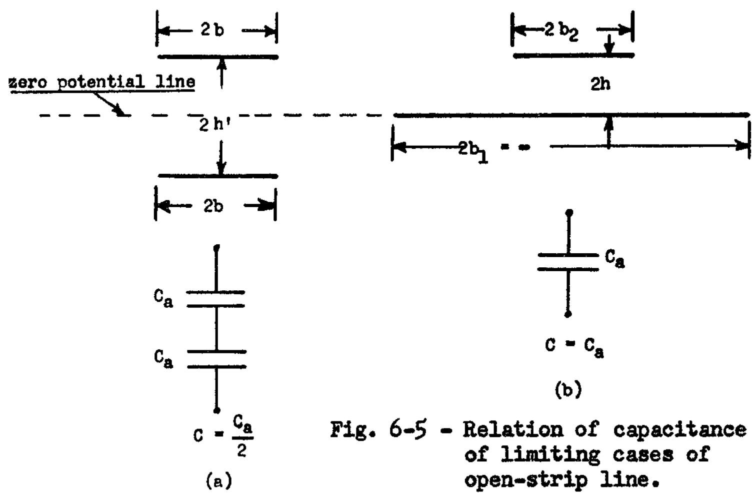

Fig. 6-5 - Relation of capacitance of limiting cases of open-strip line.

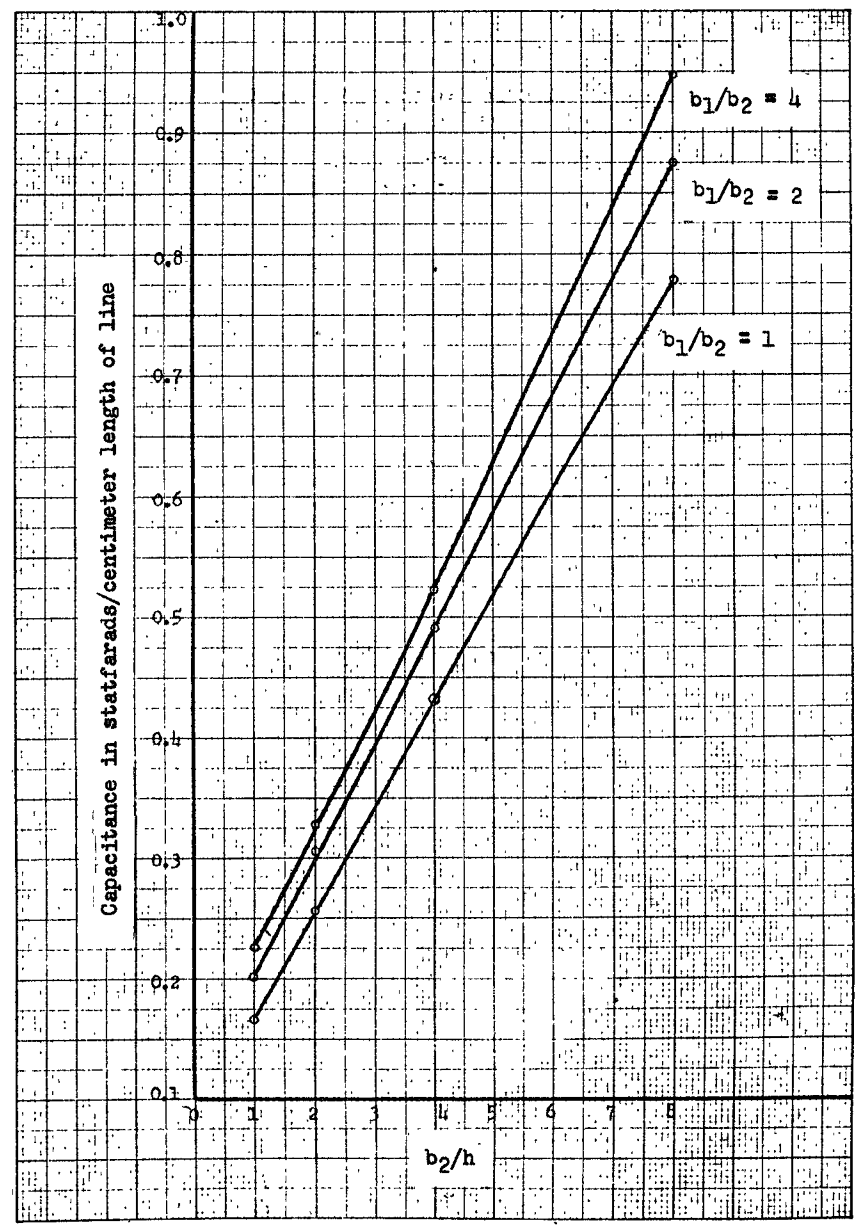

Fig. 6-4 - Universal curves for capacitance.

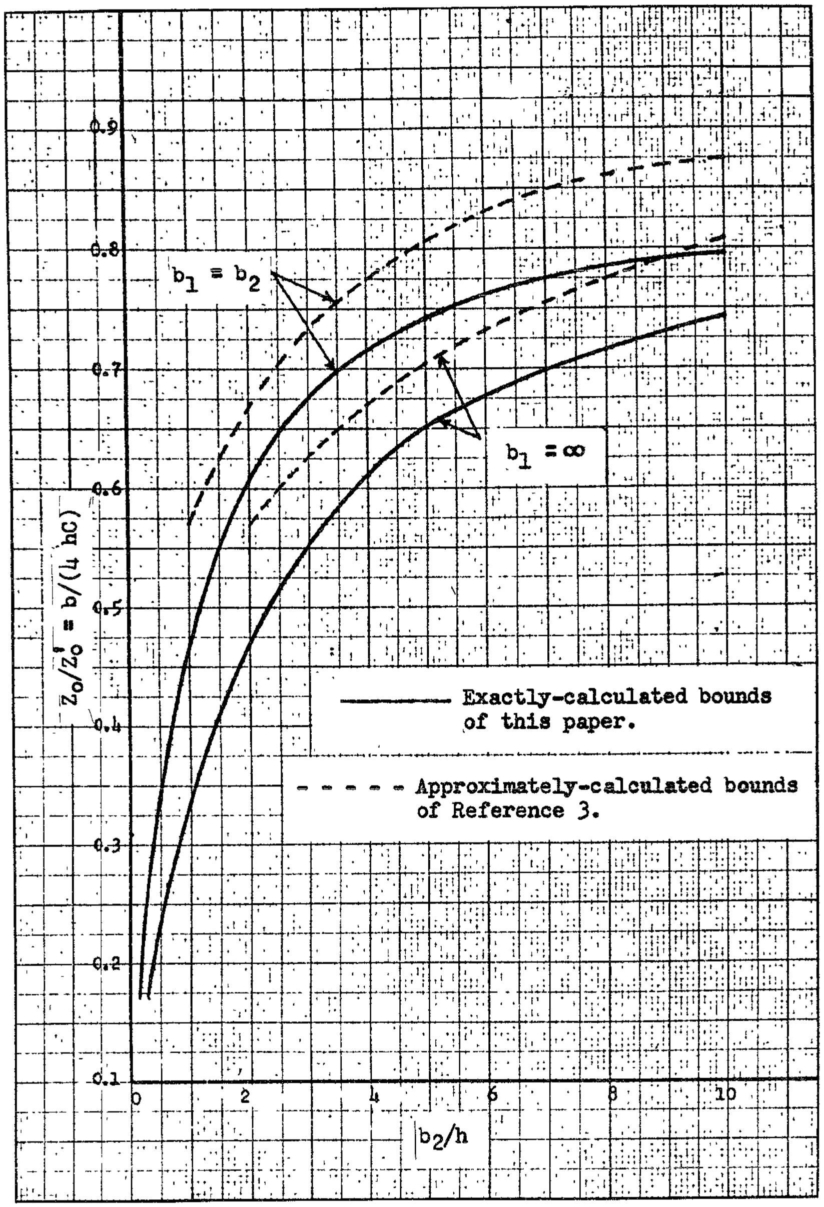

Fig. 6-6 - Comparison of exact and approximate bounds.

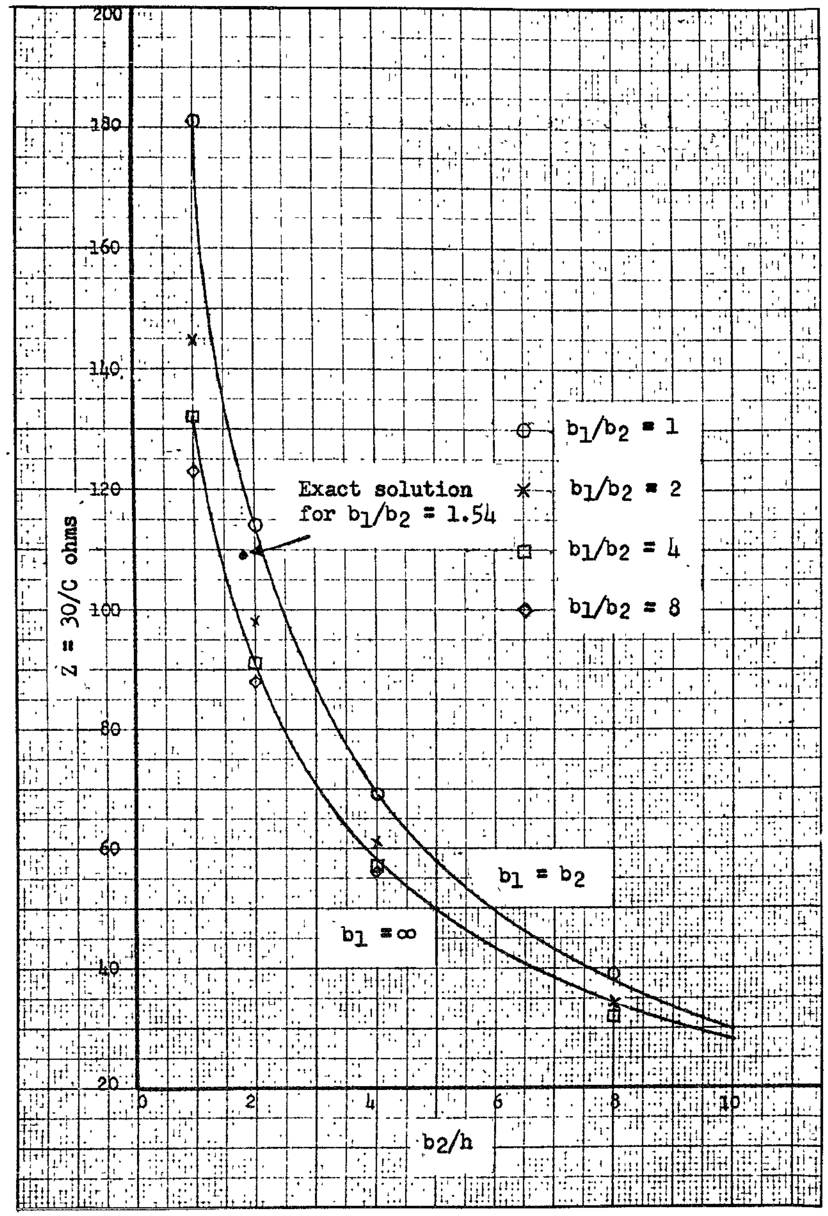

Fig. 6-7 - Characteristic impedance of various open-strip-lines.